Approximations to ideal-observer performance on signal-detection tasks

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The ideal-observer performance, as measured by the area under the receiver's operating characteristic curve, is computed for six examples of signal-detection tasks. Exact values for this quantity, as well as approximations based on the signal-to-noise ratio of the log likelihood and the likelihood-generating function, are found. The noise models considered are normal, exponential, Poisson, and two-sided exponential. The signal may affect the mean or the variance in each case. It is found that the approximation from the likelihood-generating function tracks well with the exact area, whereas the log-likelihood signal-to-noise approximation can fail badly. The signal-to-noise ratio of the likelihood ratio itself is also computed for each example to demonstrate that it is not a good measure of ideal-observer performance. © 2000 Optical Society of America

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1. Introduction

Signal detection is one of the most important tasks for imaging systems. In medical imaging, for example, we are often trying to decide whether a tumor is present or absent in a particular organ from images that are degraded by noise and have complicated anatomical backgrounds. To optimize system parameters, reconstruction methods, and image displays, we need to know how well the signal-detection task is being performed. With human observers, one normally makes this determination by setting up an observer study, which is an expensive and time-consuming process. For machine observers, we can simulate an observer study, but this can also be time consuming for realistic backgrounds and noise models.

A common way to measure the performance of an observer on a signal-detection task is to compute the area under the receiver operating characteristic (ROC) curve, to be defined below. This area is usually referred to as the area under the curve (AUC). The maximum value of the AUC, among all observers performing a given detection task and with a given noise model, is set by the ideal observer for that particular task and noise model. This is one reason that it is useful to be able to compute the performance of the ideal observer. (We describe below in detail exactly what an ideal observer is.)

We could also optimize the parameters of an imaging system if we could compute the AUC of the ideal observer that uses the raw data. An observer who is using a reconstruction algorithm and an image display cannot have a larger value for the AUC than that of the ideal observer, because any observer is, ultimately, using the raw data. The idea here is that the AUC for the ideal observer of the raw data is a measure of the ability of the data-gathering system to distinguish the signal from the background and the noise.

The ideal observer has full knowledge of the probability law on the data for both the signal-absent and the signal-present hypotheses. By computing the likelihood ratio, which is defined below, and comparing it with a threshold, the ideal observer decides whether the signal is present or absent in the data. The AUC for this observer can be difficult to compute, so it is useful to have approximations for this number. A commonly used approximation involves the signal-to-noise ratio (SNR) of the logarithm of the likelihood ratio, but this approximation, as we shall see below, can fail. Recently, we[1] introduced the concept of the likelihood-generating function for the ideal observer. This is a function of one variable that determines completely the statistical properties of the likelihood

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ratio. The likelihood-generating function, when it is evaluated at zero, gives another approximation to the AUC that is often better than the SNR approximation. Another seemingly reasonable measure of ideal-observer performance is the SNR of the likelihood ratio itself. We shall find in the examples that this statistic has peculiar properties that make it unsuitable for this purpose.

After a brief review of the main concepts to establish notation, we find the exact AUC and both approximations for a number of relevant examples. We shall see that the likelihood-generating function approximation is good even when the SNR approximation fails. Along the way, we shall examine the SNR of the likelihood ratio to see why it is not a good measure of performance. We shall also see how widely different tasks and noise models can lead to the same likelihood-generating function and therefore to the same statistics for the ideal observer. In the concluding section, we discuss how we might be able to go beyond the examples, which are rather simplistic as models of real data, and calculate approximations to the ideal-observer AUC for more realistic tasks and noise models.

2. Performance of an Observer

Any observer who makes a decision for the signal-detection task that is determined in a nonrandom way from the data vector $\mathbf{g}$ and who is not allowed to equivocate is equivalent to one who makes a decision on the basis of the value of a test statistic, the function $t(\mathbf{g})$. The vector $\mathbf{g}$ may represent a digital image, or it may be the raw data in an imaging system that performs a reconstruction to obtain such an image. After computing the test statistic for a particular data vector, the observer compares the result with a threshold t_0. If $t(\mathbf{g}) > t_0$, the observer decides that the signal is present (D_2), whereas, if $t(\mathbf{g}) < t_0$, the observer declares the signal to be absent (D_1).

A. Test Performance

Because $\mathbf{g}$ is a random vector, the statistic t is a random variable. We denote the probability density for t under the signal-absent hypothesis by $\mathrm{pr}(t|H_1)$ and the corresponding density under the signal-present hypothesis by $\mathrm{pr}(t|H_2)$. [In general, we use the notation $\mathrm{pr}(*|H_1)$ and $\mathrm{pr}(*|H_2)$ generically to denote the probability-density functions for whatever variable is in the position of the asterisk under the two hypotheses.] There are four quantities of interest when one is evaluating the performance of this observer: the true-positive fraction (TPF), the false-positive fraction (FPF), the false-negative fraction (FNF), and the true-negative fraction (TNF). These are given, respectively, by

$$\mathrm{TPF}(t_0) = \mathrm{pr}(D_2|H_2) = \int_{t_0}^{\infty} \mathrm{pr}(t|H_2)\mathrm{d}t,$$

$$\mathrm{FPF}(t_0) = \mathrm{pr}(D_2|H_1) = \int_{t_0}^{\infty} \mathrm{pr}(t|H_1)\mathrm{d}t,$$

$$\mathrm{FNF}(t_0) = \mathrm{pr}(D_1|H_2) = \int_{-\infty}^{t_0} \mathrm{pr}(t|H_2)\mathrm{d}t,$$

$$\mathrm{TNF}(t_0) = \mathrm{pr}(D_1|H_1) = \int_{-\infty}^{t_0} \mathrm{pr}(t|H_1)\mathrm{d}t. \tag{2.1}$$

These quantities satisfy the constraints that TPF + FNF = 1 and TNF + FPF = 1, so only two of them are independent. The TPF is often called the probability of detection, and the FPF, the false-alarm rate. If we know these two quantities for all t_0, we can evaluate the performance of the observer for any threshold value.

One of these numbers, or some combination of them, could be used as a figure of merit for the performance of the ideal observer, but the numbers suffer from a dependence on the threshold and the probability of occurrence of the signal. It is desirable to have a figure of merit that is independent of these quantities. The AUC is one such figure of merit.

B. Receiver Operating Characteristic Curve and the Area under the Curve

One generates the ROC curve by plotting the TPF versus the FPF, both of which vary from 0 to 1. One may also generate the curve by plotting the points [FPF(t), TPF(t)] as t varies from $-\infty$ to ∞. With this method the ROC curve begins at (1, 1) and ends at (0, 0). The area under this curve is given by

$$\mathrm{AUC}_t = \int_0^1 \mathrm{TPF}d(\mathrm{FPF}) = -\int_{-\infty}^{\infty} \mathrm{TPF}(t)\mathrm{FPF}'(t)\mathrm{d}t. \tag{2.2}$$

(The prime in the second integral indicates a derivative.) The AUC is invariant under a monotonic transformation of the decision variable, which is a useful property for a figure of merit because such a transformation produces an observer that is equivalent to the original one.

To give an alternative method for computing AUC_t, we introduce notation for expectations of a function $\omega(t)$ under the two hypotheses

$$\langle\omega\rangle_1 = \langle\omega(t)\rangle_1 = \int_{-\infty}^{\infty} \omega(t)\mathrm{pr}(t|H_1)\mathrm{d}t,$$

$$\langle\omega\rangle_2 = \langle\omega(t)\rangle_2 = \int_{-\infty}^{\infty} \omega(t)\mathrm{pr}(t|H_2)\mathrm{d}t. \tag{2.3}$$

The characteristic functions for t under the two hypotheses are then defined by

$$\psi_1(\xi) = \langle\exp(-2\pi i\xi t)\rangle_1,$$

$$\psi_2(\xi) = \langle\exp(-2\pi i\xi t)\rangle_2, \tag{2.4}$$

respectively. If these functions [Eqs. (2.4)] are known, AUC_t may be computed by means of[1]

$$\mathrm{AUC}_t = \frac{1}{2} + \frac{1}{2\pi i}\mathcal{P}\int_{-\infty}^{\infty} \psi_1(\xi)\psi_2{}^*(\xi)\,\frac{\mathrm{d}\xi}{\xi}. \tag{2.5}$$

(The letter $\mathcal{P}$ preceding the integral indicates the Cauchy principal value.) We now consider how these formulas are simplified when the observer is the ideal observer.

3. Ideal Observer

We denote the probability density for the data under the signal-absent hypothesis $\mathrm{pr}(\mathbf{g}|H_1)$; under the signal-present hypothesis it is $\mathrm{pr}(\mathbf{g}|H_2)$. The observer who maximizes the AUC among all observers for a given task and probability model uses the likelihood ratio $\Lambda(\mathbf{g})$ for the test statistic. This ratio is given by

$$\Lambda(\mathbf{g}) = \frac{\mathrm{pr}(\mathbf{g}|H_2)}{\mathrm{pr}(\mathbf{g}|H_1)}, \tag{3.1}$$

with a corresponding threshold Λ_0. Any observer who uses this or an equivalent statistic is an ideal observer. For example, an observer who uses the log likelihood $\lambda(\mathbf{g}) = \log \Lambda(\mathbf{g})$ with $\lambda_0 = \log \Lambda_0$ is also an ideal observer. This observer will make the same decision for each data vector as the one who uses $\Lambda(\mathbf{g})$ and Λ_0. If $L(\Lambda)$ is any monotonically increasing function, the observer who uses $\tilde{\lambda}(\mathbf{g}) = L[\Lambda(\mathbf{g})]$ with the threshold $\tilde{\lambda}_0 = L(\Lambda_0)$ is also an ideal observer. It is often more convenient to calculate $\tilde{\lambda}(\mathbf{g})$ that is some modification of the log likelihood than to calculate $\Lambda(\mathbf{g})$ itself.

The probability densities for likelihood ratio Λ under the two hypotheses are necessarily related by the equation $\mathrm{pr}(\Lambda|H_2) = \Lambda\mathrm{pr}(\Lambda|H_1)$.[1,2] With this relation AUC_Λ can be reduced to

$$\mathrm{AUC}_\Lambda = 1 - \frac{1}{2}\int_0^{\infty} [\mathrm{FPF}(\Lambda)]^2 \mathrm{d}\Lambda. \tag{3.2}$$

Here we provide a derivation of Eq. (3.2) because this equation is a new result. To simplify the notation, we introduce the functions $p_1(\Lambda) = \mathrm{pr}(\Lambda|H_1)$, $p_2(\Lambda) = \mathrm{pr}(\Lambda|H_2)$, and $\tilde{P}_1(\Lambda) = \mathrm{FPF}(\Lambda)$. Note that the derivative of $\tilde{P}_1(\Lambda)$ is equal to $-p_1(\Lambda)$. From the definition of AUC_Λ, we have

$$\begin{aligned}\mathrm{AUC}_\Lambda &= \int_0^{\infty} p_1(x)\left[\int_x^{\infty} p_2(y)\mathrm{d}y\right]\mathrm{d}x \\ &= \int_0^{\infty} p_1(x)\left[\int_x^{\infty} y p_1(y)\mathrm{d}y\right]\mathrm{d}x. \end{aligned} \tag{3.3}$$

The inner integral of Eq. (3.3) can be integrated by parts:

$$\int_x^{\infty} y p_1(y)\mathrm{d}y = [-y\tilde{P}_1(y)]_x^{\infty} + \int_x^{\infty} \tilde{P}_1(y)\mathrm{d}y. \tag{3.4}$$

To evaluate the first term at $y = \infty$, we have

$$y\tilde{P}_1(y) \leq \int_y^{\infty} z p_1(z)\mathrm{d}z = \int_y^{\infty} p_2(z)\mathrm{d}z. \tag{3.5}$$

This last integral in expression (3.5) vanishes as $y \to \infty$. Therefore

$$\mathrm{AUC}_\Lambda = \int_0^{\infty} x p_1(x)\tilde{P}_1(x) + \int_0^{\infty}\int_x^{\infty} p_1(x)\tilde{P}_1(y)\mathrm{d}y\mathrm{d}x. \tag{3.6}$$

The first integral in Eq. (3.6) can be integrated by parts to yield

$$\int_0^{\infty} x p_1(x)\tilde{P}_1(x)\mathrm{d}x = \left[-\frac{1}{2}x\tilde{P}_1{}^2(x)\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} \tilde{P}_1{}^2(x)\mathrm{d}x. \tag{3.7}$$

The term in square brackets in Eq. (3.7) vanishes at $x = \infty$ because both $x\tilde{P}_1(x)$ and $\tilde{P}_1(x)$ vanish in this limit. For the second integral in Eq. (3.6), we can change the order of integration:

$$\begin{aligned}\int_0^{\infty}\int_x^{\infty} p_1(x)\tilde{P}_1(y)\mathrm{d}y\mathrm{d}x &= \int_0^{\infty}\int_0^{y} p_1(x)\tilde{P}_1(y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^{\infty} [1 - \tilde{P}_1(y)]\tilde{P}_1(y)\mathrm{d}y \\ &= 1 - \int_0^{\infty} \tilde{P}_1{}^2(y)\mathrm{d}y. \end{aligned} \tag{3.8}$$

This last line in Eq. (3.8) follows from setting $x = 0$ in Eq. (3.4). Inserting Eqs. (3.7) and (3.8) into Eq. (3.6) now leads to Eq. (3.2). Thus all we need to calculate AUC_Λ is the FPF for every threshold value. This formula holds for only the likelihood ratio; it is not a general formula for computing AUC_t for an arbitrary test statistic t.

In terms of the log likelihood the relation between the probability densities under the two hypotheses is $\mathrm{pr}(\lambda|H_2) = \exp(\lambda)\mathrm{pr}(\lambda|H_1)$. By applying Eq. (2.5) to test statistic λ and using this relation between the densities, we have

$$\begin{aligned}\psi_1(\xi) &= \langle\exp(-2\pi i\xi\lambda)\rangle_1, \\ \psi_2(\xi) &= \langle\exp(-2\pi i\xi\lambda)\rangle_2 = \langle\exp(\lambda)\exp(-2\pi i\xi\lambda)\rangle_1, \end{aligned} \tag{3.9}$$

$$\mathrm{AUC}_\Lambda = \mathrm{AUC}_\lambda = \frac{1}{2} + \frac{1}{2\pi i}\,\mathcal{P}\int_{-\infty}^{\infty} \psi_1(\xi)\psi_1{}^*\left(\xi - \frac{i}{2\pi}\right)\frac{\mathrm{d}\xi}{\xi}. \tag{3.10}$$

Therefore we need to know only one of the characteristic functions for λ to calculate AUC_Λ.

Equation (3.10) may also be written in terms of the moment-generating functions for the log likelihood:

$$M_1(\beta) = \langle \exp(\beta\lambda) \rangle_1 = \langle \Lambda^\beta \rangle_1,$$
$$M_2(\beta) = \langle \exp(\beta\lambda) \rangle_2 = \langle \Lambda^\beta \rangle_2. \quad (3.11)$$

Functions (3.11) satisfy the relation $M_2(\beta) = M_1(\beta + 1)$, and AUC_Λ can be computed from

$$\text{AUC}_\Lambda = \frac{1}{2} + \frac{1}{2\pi i}\mathcal{P}\int_{-\infty}^{\infty} M_1(i\alpha)M_1(1 - i\alpha)\frac{\mathrm{d}\alpha}{\alpha}. \quad (3.12)$$

In the integral in Eq. (3.12) the line of integration in the complex plane may be shifted one-half unit to the right to yield the equivalent expression[1]

$$\text{AUC}_\Lambda = 1 - \frac{1}{4\pi}\int_{-\infty}^{\infty} M_1\left(\frac{1}{2} + i\alpha\right)M_1\left(\frac{1}{2} - i\alpha\right)\frac{\mathrm{d}\alpha}{\alpha^2 + 1/4}. \quad (3.13)$$

Finally, because of the symmetry of the integrand, Eq. (3.13) may be written as

$$\text{AUC}_\Lambda = 1 - \frac{1}{2\pi}\int_0^{\infty}\left|M_1\left(\frac{1}{2} + i\alpha\right)\right|^2\frac{\mathrm{d}\alpha}{\alpha^2 + 1/4}. \quad (3.14)$$

The integral in expression (3.14) measures how far AUC_Λ deviates from its maximum possible value of 1.

The moment-generating functions satisfy the conditions $M_1(0) = M_2(-1) = 1$ and $M_1(1) = M_2(0) = 1$. [Note that there is a typographical error in the statement of these relations above Eq. (5.5) in Ref. 1. They are stated correctly below Eq. (5.11) in that reference.] This means that they both can be written in terms of a single function, called the likelihood-generating function $G(\beta)$:

$$M_1(\beta) = \exp[\beta(\beta - 1)G(\beta - 1/2)],$$
$$M_2(\beta) = \exp[\beta(\beta + 1)G(\beta + 1/2)]. \quad (3.15)$$

We may take the first equation in this pair as the definition of the likelihood-generating function. The factor of $\beta(\beta - 1)$ assures us that $M_1(0) = M_1(1) = 1$. The second equation in the pair then follows from the fact that $M_2(\beta) = M_1(\beta + 1)$. The likelihood-generating function determines the statistics of the likelihood ratio under both hypotheses. In terms of this function AUC_Λ is given by

$$\text{AUC}_\Lambda = 1 - \frac{1}{2\pi}\int_0^{\infty}\exp\left[-2\left(\alpha^2 + \frac{1}{4}\right)\times \text{Re}G(i\alpha)\right]\frac{\mathrm{d}\alpha}{\alpha^2 + 1/4}. \quad (3.16)$$

From Eq. (3.16), we see that the values of the likelihood-generating function along the imaginary axis determine the value of AUC_Λ.

4. Signal-to-Noise Ratios

Often, the AUC_Λ is converted into a SNR by means of the inverse error function:

$$\text{SNR}_{\text{AUC}} = 2\,\text{erf}^{-1}(2\text{AUC}_\Lambda - 1). \quad (4.1)$$

Note that this quantity is unchanged if Λ is replaced with $\tilde{\lambda} = L(\Lambda)$ for monotonic L. Because it is usually difficult to compute AUC_Λ, the more typical SNR that is computed for the ideal observer is

$$\text{SNR}_\lambda = \sqrt{2}\,\frac{\langle\lambda\rangle_2 - \langle\lambda\rangle_1}{[\text{var}_1(\lambda) + \text{var}_2(\lambda)]^{1/2}}. \quad (4.2)$$

The numerator of the above fraction is the difference between the mean values of λ under the signal-present and the signal-absent hypotheses. Intuitively, this quantity is a measure of the strength of the signal in the decision statistic λ. To obtain the SNR_λ, one divides this numerator by the square root of the average variance of λ under the two hypotheses. The intuitive interpretation of this denominator is that it is a measure of the strength of the noise in the decision statistic λ. On the basis of our intuition, then, we would expect SNR_λ to increase without bound as the magnitude of the signal increases. As we shall see, this does not always occur.

Whereas the SNR_{AUC} is unchanged by a monotonic transformation of the decision variable Λ, SNR's defined analogously to Eq. (4.2) are not. For example, SNR_Λ can be quite different from SNR_λ. In fact, we shall see that the behavior of SNR_Λ as the signal strength increases runs counter to our intuitive notion of a SNR.

A third SNR, and one that also is unchanged by monotonic transformations of the decision variable, is given by

$$\text{SNR}_{G(0)} = [2G(0)]^{1/2}. \quad (4.3)$$

In fact, we have[1]

$$G(0) = -4\log M_1(1/2)$$
$$= -4\log\int_{-\infty}^{\infty}[\text{pr}(\Lambda|H_1)\text{pr}(\Lambda|H_2)]^{1/2}\mathrm{d}\Lambda. \quad (4.4)$$

The integral on the second line of Eq. (4.4) is invariant when Λ is replaced with $\tilde{\lambda} = L(\Lambda)$ for a monotonic function L.

The Bhattacharyya distance between two general probability-density functions $\rho_1(\mathbf{x})$ and $\rho_2(\mathbf{x})$ is defined by

$$d_B(\rho_1, \rho_2) = -\log\left\{\int[\rho_1(\mathbf{x})\rho_2(\mathbf{x})]^{1/2}\mathrm{d}\mathbf{x}\right\}. \quad (4.5)$$

This distance measure has proved useful in other contexts in probability theory. We can see that $G(0)$ is proportional to the Bhattacharya distance between $\text{pr}(\Lambda|H_1)$ and $\text{pr}(\Lambda|H_2)$.[3]

$\mathrm{SNR}_{G(0)}$ may also be computed directly from the probability densities on the data because

$$G(0) = -4 \log\langle \Lambda^{1/2}(\mathbf{g})\rangle_1 = -4 \log \int_{\mathbb{R}^M} \left[\frac{\mathrm{pr}(\mathbf{g}|H_2)}{\mathrm{pr}(\mathbf{g}|H_1)}\right]^{1/2} \mathrm{pr}(\mathbf{g}|H_1)\mathrm{d}\mathbf{g} = -4 \log \int_{\mathbb{R}^M} [\mathrm{pr}(\mathbf{g}|H_1)\mathrm{pr}(\mathbf{g}|H_2)]^{1/2}\mathrm{d}\mathbf{g}, \quad (4.6)$$

where the subscript $\mathbb{R}^M$ on the integral sign indicates that the integral is over all the M-dimensional data space. From Eq. (4.6), we can show that $G(0)$ is also invariant under an invertible transformation of the data. This relation (4.6) also shows that $G(0)$ is proportional to the Bhattacharya distance between $\mathrm{pr}(\mathbf{g}|H_1)$ and $\mathrm{pr}(\mathbf{g}|H_2)$.

The reason for defining $\mathrm{SNR}_{G(0)}$ as in Eq. (4.3) is as follows: The relation

$$\mathrm{AUC}_\Lambda = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{1}{2}\,\mathrm{SNR}_{\mathrm{AUC}}\right) \quad (4.7)$$

is exact, whereas

$$\mathrm{AUC}_\Lambda \approx \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{1}{2}\,\mathrm{SNR}_\lambda\right) \quad (4.8)$$

is often, but not always, a good approximation. There are reasons to believe that

$$\mathrm{AUC}_\Lambda \approx \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{1}{2}\,\mathrm{SNR}_{G(0)}\right) \quad (4.9)$$

is also a good approximation. In fact, as the examples below show, the SNR_λ approximation for AUC_Λ can fail badly, but we have not yet found an example in which the $\mathrm{SNR}_{G(0)}$ approximation is significantly far off.

We shall also see that the statistic SNR_Λ, which seems to be a reasonable thing to compute, is not related to AUC_Λ in any simple way. In fact, this statistic exhibits peculiar properties as the signal strength is increased. SNR_Λ is calculated easily from $M_1(\beta)$ through

$$\mathrm{SNR}_\Lambda = \sqrt{2}\,\frac{M_1(2) - 1}{[M_1(2) - 1 + M_1(3) - M_1^{\,2}(2)]^{1/2}}, \quad (4.10)$$

which can also be written in the form

$$\mathrm{SNR}_\Lambda = \sqrt{2}\,\frac{\mathrm{var}_1(\Lambda)}{[\mathrm{var}_1(\Lambda) + \mathrm{var}_2(\Lambda)]^{1/2}}. \quad (4.11)$$

In the examples it will often be convenient to provide the two variances in expression (4.11) and let the reader construct SNR_Λ from them.

We may also derive SNR_λ from $M_1(\beta)$ by using

$$\langle\lambda^k\rangle_1 = M_1^{(k)}(0), \quad \langle\lambda^k\rangle_2 = M_1^{(k)}(1). \quad (4.12)$$

[A superscript on the right-hand side of Eqs. (4.12) indicates the kth derivative; the one on the left-hand side is a power.] Relations (4.12) give us

$$\mathrm{SNR}_\lambda = \sqrt{2}\,\frac{M_1{}'(1) - M_1{}'(0)}{\{M_1{}''(0) - [M_1{}'(0)]^2 + M_1{}''(1) - [M_1{}'(1)]^2\}^{1/2}}. \quad (4.13)$$

It is often easier to calculate the moments directly rather than to compute the derivatives needed for Eq. (4.13). Equation (4.13) does show, however, that the analyses of all the figures of merit, AUC_λ, SNR_λ, $G(0)$, and SNR_Λ, for two different tasks and probability models are identical if the likelihood-generating functions—hence the moment-generating functions—have the same form for the two different ideal observers. We shall see examples of this situation below.

5. Examples

A. Example 1: Normal with the Signal in the Mean

If the data vector is normally distributed with a mean $\mathbf{b}$ and a covariance $\mathbf{K}$ when there is no signal present, we have

$$\mathrm{pr}(\mathbf{g}|H_1) = [(2\pi)^M \det(\mathbf{K})]^{-1/2} \times \exp\left[-\frac{1}{2}(\mathbf{g} - \mathbf{b})^{\mathrm{T}}\mathbf{K}^{-1}(\mathbf{g} - \mathbf{b})\right]. \quad (5.1)$$

(The superscript T indicates the transpose.) Equation (5.1) is sometimes a good assumption, and it is also often used as a default assumption when only the mean and the covariance are known. If the signal changes the mean from $\mathbf{b}$ to $\mathbf{b} + \mathbf{s}$, we have

$$\mathrm{pr}(\mathbf{g}|H_2) = \mathrm{pr}(\mathbf{g} - \mathbf{s}|H_1). \quad (5.2)$$

Ideal-observer performance in the situation represented by Eq. (5.2) is well known, and we simply summarize the results to compare them with other examples. The log likelihood is a linear function of the data:

$$\lambda(\mathbf{g}) = \mathbf{s}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{g} + \frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{b} - \frac{1}{2}(\mathbf{b} + \mathbf{s})^{\mathrm{T}}\mathbf{K}^{-1}(\mathbf{b} + \mathbf{s}). \quad (5.3)$$

By removing data-independent terms, we arrive at an equivalent statistic, the prewhitened matched filter:

$$\tilde{\lambda}(\mathbf{g}) = \mathbf{s}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{g} = (\mathbf{K}^{-1/2}\mathbf{s})^{\mathrm{T}}(\mathbf{K}^{-1/2}\mathbf{g}). \quad (5.4)$$

One reason that normal statistics are a favorite assumption is that all the figures of merit are easy to compute. Because λ and $\tilde{\lambda}$ differ by a data-independent constant, we have

$$\mathrm{SNR}_\lambda = \mathrm{SNR}_{\tilde{\lambda}} = (\mathbf{s}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{s})^{1/2}. \quad (5.5)$$

We also have

$$\mathrm{SNR}_{\mathrm{AUC}} = \mathrm{SNR}_{G(0)} = \mathrm{SNR}_{\lambda}, \tag{5.6}$$

which means that the approximate expressions for the AUC_{Λ} [expressions (4.9) and (4.10)] are all exact in this case.

The likelihood-generating function is constant in this example, indicating that λ itself is a normally distributed random variable. In general, the likelihood-generating function is constant if and only if the log likelihood is normally distributed. The moment-generating function under the signal-absent hypothesis in this example is

$$M_1(\beta) = \exp\left[\frac{1}{2}\beta(\beta - 1)\mathbf{s}^{\mathsf{T}}\mathbf{K}^{-1}\mathbf{s}\right]. \tag{5.7}$$

After some algebra, Eq. (5.7) yields

$$\mathrm{SNR}_{\Lambda} = \sqrt{2}\left[\frac{\exp(\mathbf{s}^{\mathsf{T}}\mathbf{K}^{-1}\mathbf{s}) - 1}{\exp(2\mathbf{s}^{\mathsf{T}}\mathbf{K}^{-1}\mathbf{s}) + 1}\right]^{1/2}. \tag{5.8}$$

From Eq. (5.8), we can show that $\mathrm{SNR}_{\Lambda}^2 \le \sqrt{2} - 1$ and $\mathrm{SNR}_{\Lambda} \to 0$ as $\mathbf{s} \to \infty$. The second property is especially troublesome because we expect better performance in the detection task as the signal strength increases. This property is the first indication that the SNR_{Λ} may not be a good measure of ideal-observer performance.

B. Example 2: Independent Exponential

A simplistic model for speckle noise maintains that the components of $\mathbf{g}$ are independent, exponentially distributed random variables. This model would correspond to small, widely spaced detectors that view a speckle pattern. A more realistic model would try to incorporate correlations among the components of $\mathbf{g}$, but that complication is not considered here. If the signal changes the mean data vector from $\mathbf{b}$ to $\mathbf{b} + \mathbf{s}$, we have

$$\mathrm{pr}(\mathbf{g}|H_1) = \prod_{m=1}^{M}\frac{1}{b_m}\exp\left(-\frac{g_m}{b_m}\right),$$
$$\mathrm{pr}(\mathbf{g}|H_2) = \prod_{m=1}^{M}\frac{1}{b_m + s_m}\exp\left(-\frac{g_m}{b_m + s_m}\right). \tag{5.9}$$

We assume in Eqs. (5.9) that $\mathbf{b}$ and $\mathbf{s}$ are vectors with positive components. If the signal had a component that was zero the corresponding detector would not contribute anything to the likelihood ratio. This relation means that M represents the number of detectors that are influenced by the signal. This assumption about the background and the signal is made in all the examples given below.

The log likelihood is

$$\lambda(\mathbf{g}) = \sum_{m=1}^{M}\frac{s_m g_m}{b_m(b_m + s_m)} + \sum_{m=1}^{M}\log\left(\frac{b_m}{b_m + s_m}\right). \tag{5.10}$$

Removing data-independent terms from Eq. (5.10) results in an equivalent observer that is linear in $\mathbf{g}$:

$$\tilde{\lambda}(\mathbf{g}) = \sum_{m=1}^{M}\frac{s_m g_m}{b_m(b_m + s_m)}. \tag{5.11}$$

Because the variance of an exponential random variable is the square of the mean, each component of the template that this observer uses is the corresponding component of the signal divided by the geometric mean of the variances of that data component under the two hypotheses. Therefore components with larger variances are weighted less in computing $\tilde{\lambda}(\mathbf{g})$.

If we introduce the normalized quantities $\pi_m = b_m(b_m + s_m)^{-1}$ and $\gamma_m = s_m(b_m + s_m)^{-1}$, the moment-generating function for λ when the signal is absent is

$$M_1(\beta) = \prod_{m=1}^{M}\frac{{\pi_m}^{\beta}}{1 - \gamma_m\beta}. \tag{5.12}$$

For real values of β Eq. (5.12) is valid as long as $\gamma_m\beta < 1$ for all m. For larger values of β, $M_1(\beta)$ is infinite. The expression given here for $M_1(\beta)$ may be understood as the analytic continuation of $\langle\Lambda^{\beta}\rangle_1$ in regions of the complex β plane, where the integral in this expectation does not converge absolutely.

The exact value for the AUC_{Λ} can be computed from Eq. (3.13) by use of contour integration. If we introduce inverse contrast ratios $\rho_m = b_m {s_m}^{-1}$ and, as a notational convenience, define $\rho_0 = 0$, and if these numbers are all distinct, the integrand has only simple poles, and the integral can be evaluated easily. The result is

$$\mathrm{AUC}_{\lambda} = 1 - \frac{1}{2}A_1A_2, \tag{5.13}$$

with

$$A_1 = \prod_{m=1}^{M}\rho_m(1 + \rho_m), \tag{5.14}$$

$$A_2 = \sum_{n=0}^{M}\left[(1 + 2\rho_n)\prod_{\substack{m=0\\ m\ne n}}^{M}(1 + \rho_m + \rho_n)(\rho_m - \rho_n)\right]^{-1}. \tag{5.15}$$

From Eqs. (5.13)–(5.15), we can see that $\mathrm{AUC}_{\Lambda} \to 1$ as $\mathbf{s} \to \infty$. This relation implies that $\mathrm{SNR}_{\mathrm{AUC}} \to \infty$ as $\mathbf{s} \to \infty$.

The log-likelihood SNR, SNR_{λ}, is given by

$$\mathrm{SNR}_{\lambda} = \sqrt{2}\left[\sum_{m=1}^{M}\frac{1}{\rho_m(\rho_m + 1)}\right] \times \left[\sum_{m=1}^{M}\left(\frac{1}{\rho_m + 1}\right)^2 + \left(\frac{1}{\rho_m}\right)^2\right]^{-1/2}. \tag{5.16}$$

For this SNR we can use the Schwartz inequality on the numerator to show that

$$\mathrm{SNR}_{\lambda} \le \left[2\sum_{m=1}^{M}\left(\frac{1}{\rho_m + 1}\right)^2\right]^{1/2} \le \sqrt{2M}. \tag{5.17}$$

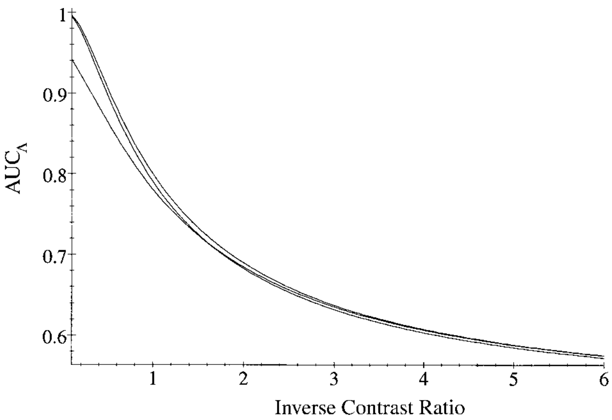

Fig. 1. Exact AUC_Λ and the approximations to AUC_Λ that are derived from SNR_λ and $SNR_{G(0)}$ for example 2 (independent exponential noise) with $M = 3$. These AUC values are plotted versus the inverse contrast ratio (background to signal). On the left-hand side of the graph the curve that represents the exact value is below the curve for $SNR_{G(0)}$ and above the curve for SNR_λ. The curve for the exact value is below the curves for both approximations on the right-hand side of the graph.

Relation (5.17) means that the SNR_λ approximation for AUC_Λ must fail badly for strong signals. Of course, we usually are interested in performance with weak signals, so this failure may not be a problem in practice.

From the expression for $M_1(\beta)$ [Eqs. (4.12)], we find that

$$SNR_{G(0)} = 2\left\{\sum_{m=1}^{M} \log\left[\frac{(2\rho_m + 1)^2}{4\rho_m(\rho_m + 1)}\right]\right\}^{1/2}. \quad (5.18)$$

From Eq. (5.18), we see that $SNR_{G(0)} \to \infty$ as $\mathbf{s} \to \infty$.

To compare the AUC_Λ approximations graphically, we consider a special case in which $\rho_1 = \ldots = \rho_m = \rho$. In this case it is actually easier to use Eq. (3.12) to compute the exact value for AUC_Λ. After we use the partial fraction expansion

$$\frac{1}{\alpha(1 - i\gamma\alpha)^M} = \frac{1}{\alpha} + \frac{i\gamma}{1 - i\gamma\alpha} + \cdots + \frac{i\gamma}{(1 - i\gamma\alpha)^M} \quad (5.19)$$

and the contour integration, the result is

$$AUC_\Lambda = \left(\frac{1 + \rho}{1 + 2\rho}\right)^M \sum_{k=0}^{M-1} \frac{(M + k - 1)!}{k!(M - 1)!} \left(\frac{\rho}{1 + 2\rho}\right)^k, \quad (5.20)$$

which we compare in Fig. 1 with the SNR_λ approximation

$$AUC_\lambda \approx \frac{1}{2} + \frac{1}{2}\,\text{erf}\left[\frac{1}{2}\left(\frac{2M}{2\rho^2 + 2\rho + 1}\right)^{1/2}\right] \quad (5.21)$$

and the $SNR_{G(0)}$ approximation

$$AUC_\lambda \approx \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\left\{M \log\left[\frac{(2\rho + 1)^2}{4\rho(\rho + 1)}\right]\right\}^{1/2}\right) \quad (5.22)$$

for $M = 3$. Note that a strong signal corresponds to a small ρ value. As M increases, the $SNR_{G(0)}$ approximation improves. The SNR_λ approximation must always fail as $\rho \to 0$.

It is easiest to express SNR_Λ in terms of the π_m variables. The relevant variances are

$$\text{var}_1(\Lambda) = \prod_{m=1}^{M} \frac{{\pi_m}^2}{2\pi_m - 1} - 1, \quad (5.23)$$

$$\text{var}_2(\Lambda) = \prod_{m=1}^{M} \frac{{\pi_m}^3}{3\pi_m - 2} - \prod_{m=1}^{M} \frac{{\pi_m}^4}{(2\pi_m - 1)^2}. \quad (5.24)$$

Because of the limitations on β that were noted above, the expression for $\text{var}_2(\Lambda)$ is valid only if all $\pi_m > 2/3$. If any value of π_m does not satisfy this condition SNR_Λ is not defined. As $s_m \to \infty$, we will have $\pi_m \to 0$. Therefore SNR_Λ eventually becomes undefined as we increase the signal strength. As the smallest value of π_m approaches $2/3$, we see that $SNR_\Lambda \to 0$. Again, this peculiar behavior makes SNR_Λ an unsuitable measure of ideal-observer performance.

C. Example 3: Poisson Statistics

Poisson statistics are important in imaging problems for which the number of photons collected is relatively small. If the signal changes the mean from $\mathbf{b}$ to $\mathbf{b} + \mathbf{s}$, we have

$$\text{pr}(\mathbf{g}|H_1) = \prod_{m=1}^{M} \frac{\exp[g_m \log(b_m) - b_m]}{g_m!},$$
$$\text{pr}(\mathbf{g}|H_2) = \prod_{m=1}^{M} \frac{\exp[g_m \log(b_m + s_m) - b_m - s_m]}{g_m!}. \quad (5.25)$$

Again, we assume in Eqs. (5.25) that both $\mathbf{b}$ and $\mathbf{s}$ are positive vectors. If we let $\kappa_m = (b_m + s_m){b_m}^{-1}$, the log likelihood is

$$\lambda(\mathbf{g}) = \sum_{m=1}^{M} g_m \log \kappa_m - \sum_{m=1}^{M} s_m, \quad (5.26)$$

and an equivalent observer that is linear in $\mathbf{g}$ is given by[4]

$$\tilde{\lambda}(\mathbf{g}) = \sum_{m=1}^{M} g_m \log \kappa_m. \quad (5.27)$$

The moment-generating function for λ under the signal-absent hypothesis is

$$M_1(\beta) = \exp\left(-\beta \sum_{m=1}^{M} s_m\right)\exp\left[\sum_{m=1}^{M} b_m({\kappa_m}^\beta - 1)\right]. \quad (5.28)$$

From Eq. (5.28) the $SNR_{G(0)}$ can be computed as

$$SNR_{G(0)} = 2\left(2\left\{\sum_{m=1}^{M} \frac{1}{2}(2b_m + s_m) - [b_m(b_m + s_m)]^{1/2}\right\}\right)^{1/2}$$
$$= 2\left\{\sum_{m=1}^{M} [(b_m + s_m)^{1/2} - (b_m)^{1/2}]^2\right\}^{1/2}, \quad (5.29)$$

where the quantity in braces in the first square root is the sum of the differences between the arithmetic means of b_m and $b_m + s_m$ and their geometric means. The SNR_λ is given by

$$\mathrm{SNR}_\lambda = \sqrt{2}\left(\sum_{m=1}^{M} s_m \log \kappa_m\right) \times \left\{\left[\sum_{m=1}^{M} (2b_m + s_m)\log^2 \kappa_m\right]^{1/2}\right\}^{-1}. \quad (5.30)$$

The exact value for the AUC_Λ can be worked out in the general case, but we restrict ourselves to the special case for which $b_1 = \ldots = b_m = b$ and $s_1 = \ldots = s_m = s$. The integral in Eq. (3.14) in this case reduces to

$$\frac{1}{2\pi}\exp[-M(2b+s)]\int_0^\infty \exp\left\{M[b(b+s)]^{1/2}\cos\left[\alpha \log\left(\frac{b+s}{b}\right)\right]\right\}\frac{d\alpha}{\alpha^2 + 1/4}. \quad (5.31)$$

To execute this integral, we consider a more general situation. Suppose that a function $f(r)$ has an expansion

$$f(r) = \sum_{k=0}^{\infty} f_k r^k \quad (5.32)$$

that converges everywhere. Then there is also an expansion

$$f(x+y) = \sum_{k \neq l}^{\infty} f_{kl}(x^k y^l + x^l y^k) + \sum_{k=0}^{\infty} f_{kk} x^k y^k, \quad (5.33)$$

with

$$f_{kl} = \binom{k+l}{l} f_{k+l}. \quad (5.34)$$

By using this expansion [Eq. (5.33)] for the integrand and implementing the contour integration term by term, we can show that

$$\frac{1}{2\pi}\int_0^\infty f\left[2\sqrt{xy}\cos\left(\alpha \ln\frac{x}{y}\right)\right]\frac{d\alpha}{\alpha^2 + 1/4} = \sum_{k\neq l} f_{kl}\min\{x^k y^l, x^l y^k\} + \frac{1}{2}\sum_k f_{kk} x^k y^k. \quad (5.35)$$

If we now set $x = b$, $y = (b + s)$, and $f(r) = \exp(Mr)$, we find that

$$\mathrm{AUC}_\Lambda = 1 - \exp[-M(2b+s)] \times \left[\sum_{k=0}^{\infty}\sum_{l=0}^{k-1}\frac{M^{k+l} b^k (b+s)^l}{k!\,l!} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{M^{2k} b^k (b+s)^k}{k!\,k!}\right]. \quad (5.36)$$

In the general case, we can use the power-series expansion of $f(x_1 + y_1 + \ldots + x_M + y_M)$ to get a similar result.

In Fig. 2, we compare the exact AUC_Λ for $M_b = 0.1$ with the SNR_λ approximation

$$\mathrm{AUC}_\Lambda \approx \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left[\left(\frac{M}{2}\right)^{1/2}\frac{s}{(2b+s)^{1/2}}\right] \quad (5.37)$$

and the $\mathrm{SNR}_{G(0)}$ approximation

$$\mathrm{AUC}_\Lambda \approx \frac{1}{2} + \frac{1}{2}\operatorname{erf}[\sqrt{M}\,(\sqrt{b+s} - \sqrt{b})] \quad (5.38)$$

as functions of Ms. It is clear from Fig. 2 that

$$\mathrm{SNR}_{G(0)} \geq \mathrm{SNR}_{\mathrm{AUC}}, \qquad \mathrm{SNR}_{G(0)} \geq \mathrm{SNR}_\lambda. \quad (5.39)$$

Relations (5.39) are, in fact, true for the general signal and the general background as well in this example. For values of Mb greater than 1, the three curves are almost indistinguishable.

For SNR_Λ, we get the variances

$$\mathrm{var}_1(\Lambda) = \prod_{m=1}^{M}\exp\left(\frac{{s_m}^2}{b_m}\right) - 1, \quad (5.40)$$

$$\mathrm{var}_2(\Lambda) = \prod_{m=1}^{M}\exp[s_m({\kappa_m}^2 + \kappa_m - 2)] - \prod_{m=1}^{M}\exp\left(\frac{2{s_m}^2}{b_m}\right). \quad (5.41)$$

Once again, we have $\mathrm{SNR}_\Lambda \to 0$ as the signal strength increases.

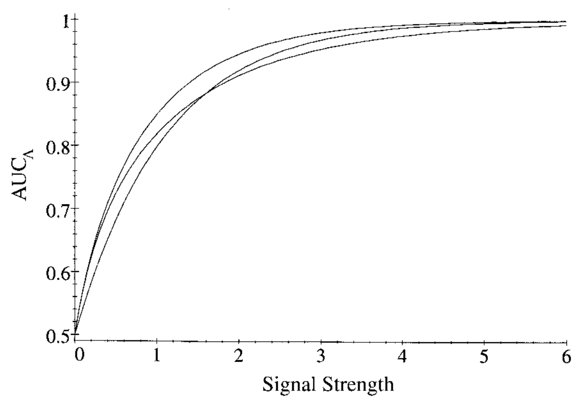

Fig. 2. Exact AUC_λ and the approximations to this value that are derived from SNR_λ and $\mathrm{SNR}_{G(0)}$ for example 3 (Poisson noise) with $Mb = 0.1$. These AUC values are plotted versus the signal strength, which represents the mean number of photons from the signal in the data. On the left-hand side of the graph the curve for the exact value is below the curve for the SNR_λ approximation, which in turn is below the curve for the $\mathrm{SNR}_{G(0)}$ approximation. The curve for the exact value is between the curves for the two approximations on the right-hand side.

D. Example 4: Independent Two-Sided Exponential with the Signal in the Mean

There is some evidence that, when natural backgrounds are convolved with a high-pass filter, the gray-level histograms of the resultant images have the shape of a two-sided exponential density function.[5] If the signal passes through the filter and we again make the simplistic assumption that the pixels in the filtered image are independent, the probability-density functions for the data are

$$\mathrm{pr}(\mathbf{g}|H_1) = \prod_{m=1}^{M} \frac{1}{2c_m} \exp\left(-\frac{1}{c_m}|g_m - b_m|\right),$$

$$\mathrm{pr}(\mathbf{g}|H_2) = \prod_{m=1}^{M} \frac{1}{2c_m} \exp\left(-\frac{1}{c_m}|g_m - b_m - s_m|\right). \quad (5.42)$$

As with speckle noise, a model that takes correlations between pixel values into account would be more realistic, but we do not attempt that calculation here. The log likelihood is given by

$$\lambda(\mathbf{g}) = \sum_{m=1}^{M} \frac{1}{c_m} (|g_m - b_m| - |g_m - b_m - s_m|). \quad (5.43)$$

Unlike in the previous examples this is a nonlinear function of the data.

If we introduce $\theta_m = |s_m| c_m{}^{-1}$, the moment-generating function for λ when the signal is absent is

$$M_1(\beta) = \prod_{m=1}^{M} \frac{1}{2}\left\{1 - \frac{1}{2\beta - 1} + \left(\frac{2\beta}{2\beta - 1}\right) \times \exp[(2\beta - 1)\theta_m]\right\}\exp(-\theta_m\beta), \quad (5.44)$$

which leads directly to

$$\mathrm{SNR}_{G(0)} = 2\left\{\sum_{m=1}^{M}\left[\theta_m - 2\log\left(1 + \frac{\theta_m}{2}\right)\right]\right\}^{1/2}. \quad (5.45)$$

To compute SNR_λ, it is easier to use the modified log likelihood:

$$\tilde{\lambda}(\mathbf{g}) = \lambda(\mathbf{g}) + \sum_{m=1}^{M} \theta_m. \quad (5.46)$$

Note that $\mathrm{SNR}_\lambda = \mathrm{SNR}_{\tilde{\lambda}}$. Inasmuch as this function $\tilde{\lambda}(\mathbf{g})$ is still nonlinear in the data, $\mathrm{SNR}_{\tilde{\lambda}}$ is somewhat complicated. For this reason, we show each component of $\mathrm{SNR}_{\tilde{\lambda}}$ individually:

$$\langle\tilde{\lambda}\rangle_1 = \sum_{m=1}^{M} [1 - \exp(-\theta_m)],$$
$$\langle\tilde{\lambda}\rangle_2 = \sum_{m=1}^{M} [2\theta_m - 1 + \exp(-\theta_m)], \quad (5.47)$$

$$\mathrm{var}_1(\tilde{\lambda}) = \sum_{m=1}^{M} [3 - 2\exp(-\theta_m) - 4\theta_m \exp(-\theta_m) - \exp(-2\theta_m)] = \mathrm{var}_2(\tilde{\lambda}). \quad (5.48)$$

The exact AUC_Λ is given by

$$\mathrm{AUC}_\Lambda = 1 - \frac{1}{2\pi}\left(\frac{1}{2}\right)^{2M} \exp\left(-\sum_{m=1}^{M}\theta_m\right)\left\{\int_0^\infty \left(\frac{1}{\alpha}\right)^{2M} \prod_{m=1}^{M} [\sin(\alpha\theta_m) + 2\alpha\cos(\alpha\theta_m)]^2 \frac{\mathrm{d}\alpha}{\alpha^2 + 1/4}\right\}. \quad (5.49)$$

Unfortunately, the integral in expression (5.49) resists contour-integration methods. The case of $M = 1$ can be computed with Eq. (3.2), and the result is

$$\mathrm{AUC}_\Lambda = 1 - \frac{1}{2}\exp(-\theta)\left(1 + \frac{\theta}{2}\right). \quad (5.50)$$

In Fig. 3 this exact expression (5.50) is compared with the SNR_λ approximation

$$\mathrm{AUC}_\Lambda \approx \frac{1}{2} + \frac{1}{2}\,\mathrm{erf} \times \left\{\frac{\theta - 1 + \exp(-\theta)}{[3 - (2 + 4\theta)\exp(-\theta) - \exp(-2\theta)]^{1/2}}\right\} \quad (5.51)$$

and the $\mathrm{SNR}_{G(0)}$ approximation

$$\mathrm{AUC}_\Lambda \approx \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left\{\left[\theta - 2\log\left(1 + \frac{\theta}{2}\right)\right]^{1/2}\right\}. \quad (5.52)$$

The SNR_Λ in this example can be computed from

$$\mathrm{var}_1(\Lambda) = \prod_{m=1}^{M} \frac{1}{3}[\exp(-2\theta_m) + 2\exp(\theta_m)] - 1, \quad (5.53)$$

$$\mathrm{var}_2(\Lambda) = \prod_{m=1}^{M} \frac{1}{5}[2\exp(-3\theta_m) + 3\exp(2\theta_m)] - \prod_{m=1}^{M} \frac{1}{9}[\exp(-2\theta_m) + 2\exp(\theta_m)]^2. \quad (5.54)$$

For strong signals and large M, we have $\mathrm{SNR}_\Lambda \approx \sqrt{2}(\sqrt{3/5})^M$, which is a small number. In this case not only the small value but also the dependence on the number of detectors influenced by the signal is counterintuitive.

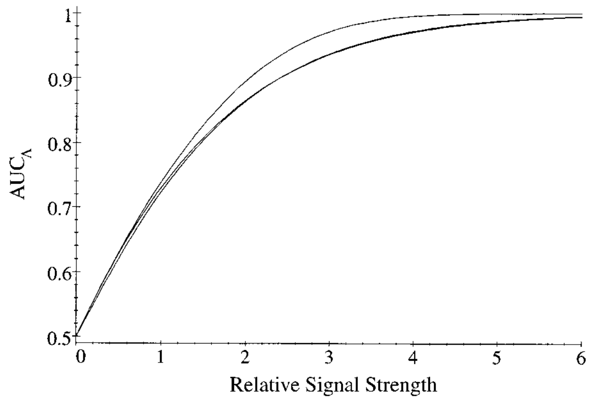

Fig. 3. Exact AUC_Λ and the approximations to this number that are derived from SNR_λ and $\mathrm{SNR}_{G(0)}$ for example 4 (Laplacian noise) with $M = 1$. These AUC values are plotted versus the relative signal strength (signal/noise level). On the left-hand side of the graph the curve for the exact value is below the curve for the $\mathrm{SNR}_{G(0)}$ approximation, which in turn is below the curve for the SNR_λ approximation. The curve for the exact value is between the curves for the two approximations on the right-hand side.

E. Example 5: Independent Two-Sided Exponential with the Signal in the Variance

The signal in a filtered image may not be in the mean. If the filtered images are uniform zero-mean textures under both hypotheses they still may differ in the variance. The signal in this case is a perhaps subtle change in the statistical properties of the image. When the signal is in the variance for independent two-sided exponential data, we have

$$\mathrm{pr}(\mathbf{g}|H_1) = \prod_{m=1}^{M} \frac{1}{2c_m} \exp\left(-\frac{1}{c_m}|g_m - b_m|\right),$$
$$\mathrm{pr}(\mathbf{g}|H_2) = \prod_{m=1}^{M} \frac{1}{2s_m c_m} \exp\left(-\frac{1}{s_m c_m}|g_m - b_m|\right). \quad (5.55)$$

The corresponding log likelihood is given by

$$\lambda(\mathbf{g}) = \sum_{m=1}^{M} \frac{1}{\beta_m}\left(\frac{s_m - 1}{s_m}\right)|g_m - b_m| - \sum_{m=1}^{M} \log(s_m), \quad (5.56)$$

which is again a nonlinear function of the data.

If we set $\tau_m = (s_m - 1)s_m^{-1}$, the moment-generating function for λ under the signal-absent hypothesis is

$$M_1(\beta) = \prod_{m=1}^{M} \frac{s_m^{-\beta}}{1 - \tau_m\beta}. \quad (5.57)$$

By making the identifications $\pi_m = s_m^{-1}$, $\gamma_m = (s_m - 1)s_m^{-1}$, and $\rho_m = (s_m - 1)^{-1}$, we see that Eq. (5.57) is the same moment-generating function that we had in example 2. The quantities AUC_λ, SNR_λ, SNR_Λ, and $G(0)$ can be taken directly from that example.

F. Example 6: Normal with the Signal in the Variance

The signal may also appear in the covariance matrix with normal noise. This would occur if the signal corresponded to a change in the texture and if the texture could be described by a normal random process. In this case the density functions are

$$\mathrm{pr}(\mathbf{g}|H_1) = [(2\pi)^M \det(\mathbf{K}_1)]^{-1/2}\exp\left[-\frac{1}{2}(\mathbf{g} - \mathbf{b})^\mathrm{T}\mathbf{K}_1^{-1}(\mathbf{g} - \mathbf{b})\right],$$
$$\mathrm{pr}(\mathbf{g}|H_2) = [(2\pi)^M \det(\mathbf{K}_2)]^{-1/2}\exp\left[-\frac{1}{2}(\mathbf{g} - \mathbf{b})^\mathrm{T}\mathbf{K}_2^{-1}(\mathbf{g} - \mathbf{b})\right]. \quad (5.58)$$

The log likelihood is a quadratic function of the data:

$$\lambda(\mathbf{g}) = \frac{1}{2}\mathbf{g}^\mathrm{T}(\mathbf{K}_1^{-1} - \mathbf{K}_2^{-1})\mathbf{g} + \frac{1}{2}\log[\det(\mathbf{K}_1\mathbf{K}_2^{-1})]. \quad (5.59)$$

This statistic [Eq. (5.59)] essentially computes the difference between the squared magnitudes of the prewhitened data vectors.

The moment-generating function for λ when the signal is absent is given by

$$M_1(\beta) = \frac{\{[\det(\mathbf{K}_1)]^{1/2}\}^{\beta-1}\{[\det(\mathbf{K}_2)]^{1/2}\}^{-\beta}}{\{\det[(1-\beta)\mathbf{K}_1^{-1} + \beta\mathbf{K}_2^{-1}]\}^{1/2}}. \quad (5.60)$$

If we use the identity $\mathbf{K}_1^{-1} + \mathbf{K}_2^{-1} = \mathbf{K}_1^{-1}(\mathbf{K}_2 + \mathbf{K}_1)\mathbf{K}_2^{-1}$, $\mathrm{SNR}_{G(0)}$ reduces to

$$\mathrm{SNR}_{G(0)} = \sqrt{2}\,\{2\log[\det(\mathbf{K}_1 + \mathbf{K}_2)] - \log[\det(\mathbf{K}_1)] - \log[\det(\mathbf{K}_2)] - 2M\log 2\}^{1/2}. \quad (5.61)$$

Note that we do not need to invert the covariance matrices to compute this quantity, even though we need those inverses to find the log likelihood.

For the exact AUC_Λ, we can start with Eq. (3.13) and make the change of variables

$$z = \frac{2i\alpha - 1}{2i\alpha + 1} \quad (5.62)$$

to arrive at an integral near the unit circle C:

$$\mathrm{AUC}_\Lambda = 1 - \frac{1}{4\pi i} \times \oint_C \frac{(1-z)^M}{\{\det[(\mathbf{K}_1\mathbf{K}_2^{-1} - z\mathbf{I})(\mathbf{K}_2\mathbf{K}_1^{-1} - z\mathbf{I})]\}^{1/2}z}. \quad (5.63)$$

Because $\mathbf{K}_1\mathbf{K}_2^{-1} = (\mathbf{K}_2\mathbf{K}_1^{-1})^{-1}$, there are exactly $M + 1$ poles inside C, counting multiplicities.

The square root complicates evaluation of the integral in Eq. (5.63). For the special case of $\mathbf{K}_2 = a\mathbf{K}_1$ it can be computed by contour integration when M is even. In fact, if $M = 2N$, the moment-generating function in this special case is

$$M_1(\beta) = \left\{ \frac{a^{-\beta}}{1 + [(1/a) - 1]\beta} \right\}^N. \qquad (5.64)$$

By setting (for all n) the values $\pi_n = a^{-1}$, $\gamma_n = (a - 1)a^{-1}$, and $\rho_n = (a - 1)^{-1}$, we can identify this moment-generating function as the one from example 2. We do not derive SNR_λ for the general case in this example, although for the special case it could be found from the SNR_λ in example 2.

6. Conclusions

The main conclusion that we can draw from these examples is that $\mathrm{SNR}_{G(0)}$ has certain advantages compared with SNR_λ as a measure of performance of the ideal observer. It often gives a better approximation to the AUC_Λ through the error-function relation. $\mathrm{SNR}_{G(0)}$ is also invariant under monotonic transformations of the decision statistic and invertible transformations of the data. However, we have seen that SNR_Λ is not a good choice for representing performance of the ideal observer. This statistic does not have the kind of dependence on signal strength that we would expect. For examples of this behavior in other contexts see Refs. 6 and 7.

For an example of how the invariance properties of $\mathrm{SNR}_{G(0)}$ could be useful, suppose that we have an invertible transformation $\tilde{\mathbf{g}} = \Gamma(\mathbf{g})$ and that the components of $\tilde{\mathbf{g}}$ are approximately statistically independent under both the signal-absent and the signal-present hypotheses. With a weak signal this independence should be possible, and it may also be possible with stronger signals. Such a transformation Γ could be the outcome of independent-components analysis of the data, for example. We would then have

$$\exp\left[-\frac{1}{4}G(0)\right] \approx \prod_{m=1}^{M} \int [\mathrm{pr}(\tilde{g}_m|H_1)\mathrm{pr}(\tilde{g}_m|H_2)\mathrm{d}g_m]^{1/2}. \qquad (6.1)$$

The one-dimensional integrals could be computed numerically. Another option would be to model the components of $\tilde{\mathbf{g}}$ as random variables of the types described in the examples in Section 4 and then to get values for these integrals from the cases of $M = 1$ in those calculations.

The independent components may also reveal how to modify a system to produce better performance for a particular task. If some combination of parameter changes in the system decreases the value of one of the integrals in relation (6.1) without changing the others much, the overall performance is increased, as measured by this approximation to $G(0)$. We hope to implement these ideas in the near future for actual imaging systems.

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