Bounds on the area under the receiver operating characteristic curve for the ideal observer

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A new upper bound is derived on the area under the receiver operating characteristic curve for the ideal observer in a signal-detection task. This upper bound is determined by the values of the likelihood-generating function and its second derivative at the origin. This bound is compared with other bounds on ideal-observer performance that have been derived recently, and it is also shown how this bound leads to some asymptotic results for approximations to ideal-observer performance. © 2002 Optical Society of America

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1. INTRODUCTION

The receiver operating characteristic (ROC) curve is often used to quantify the performance of an observer on a signal-detection task. The area under the ROC curve (AUC) is a scalar that can be computed or estimated from ROC data and used as a figure of merit for observer performance on the task in question. Given a task and a statistical model of the data, the ideal observer computes a statistic called the likelihood ratio and compares it with a threshold in order to decide whether the signal is present or absent. The AUC of the ideal observer is greater than or equal to the AUC for any other observer for a given detection task and therefore can be used as a measure of the quality of the data for the task. This in turn implies that the ideal-observer AUC can be used as a figure of merit for the imaging system that generates the data.

Unfortunately, the AUC for the ideal observer can be difficult to compute for realistic imaging system models and detection tasks. There has been some interest recently in finding upper and lower bounds for the ideal-observer AUC, or other figures of merit derived from the ROC curve, in terms of quantities that may be easier to compute.[1–4] Some of these bounds will be summarized, and then a new upper bound on the ideal-observer AUC will be derived. These upper and lower bounds will be compared in two examples where the ideal-observer AUC is known exactly, and they will also be compared with an approximation for the ideal-observer AUC that is given in Ref. 5. Finally, it will be shown how the new upper bound leads to some asymptotic results on the ideal-observer AUC when the data are independent and identically distributed.

2. IDEAL-OBSERVER AREA UNDER THE CURVE

The ideal observer takes the data vector $\mathbf{g}$ and computes the likelihood ratio

$$\Lambda(\mathbf{g}) = \frac{\mathrm{pr}(\mathbf{g}|H_1)}{\mathrm{pr}(\mathbf{g}|H_0)}. \tag{1}$$

The numerator is the probability density for the data under the signal-present hypothesis, and the denominator is the density for the signal-absent hypothesis. This statistic is compared with a threshold Λ_0, and the signal is declared to be present if $\Lambda(\mathbf{g}) > \Lambda_0$; otherwise, it is declared to be absent.

The ROC curve for the ideal observer is a plot of the true-positive fraction (TPF)

$$\mathrm{TPF}(\Lambda_0) = \int_{\Lambda_0}^{\infty} \mathrm{pr}(\Lambda|H_1)\mathrm{d}\Lambda \tag{2}$$

versus the false-positive fraction (FPF)

$$\mathrm{FPF}(\Lambda_0) = \int_{\Lambda_0}^{\infty} \mathrm{pr}(\Lambda|H_0)\mathrm{d}\Lambda \tag{3}$$

as the threshold is varied from 0 to ∞. This curve starts at the point (1, 1) and ends at (0, 0). It lies above the ROC curve for any other test statistic for the given detection task. The AUC for the ideal observer is given by several equivalent expressions, some of which are given below:

$$\begin{aligned}\mathrm{AUC}_\Lambda &= \int_0^1 \mathrm{TPF}\mathrm{d}(\mathrm{FPF})\\ &= \int_0^\infty \mathrm{TPF}(\Lambda)\mathrm{pr}(\Lambda|H_0)\mathrm{d}\Lambda\\ &= \int_0^\infty \int_\Lambda^\infty \mathrm{pr}(\Lambda'|H_1)\mathrm{pr}(\Lambda|H_0)\mathrm{d}\Lambda'\mathrm{d}\Lambda\\ &= 1 - \int_0^\infty \int_0^\Lambda \mathrm{pr}(\Lambda'|H_1)\mathrm{pr}(\Lambda|H_0)\mathrm{d}\Lambda'\mathrm{d}\Lambda. \end{aligned} \tag{4}$$

We will use the last expression in this list to derive the new upper bound on AUC_Λ.

The moment-generating function for Λ under the hypothesis H_0 is given by

$$M_0(\beta) = \int_0^\infty \Lambda^\beta \mathrm{pr}(\Lambda|H_0)\mathrm{d}\Lambda = \langle \Lambda^\beta \rangle_0. \tag{5}$$

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If this function is known, then the statistics of the likelihood ratio are completely determined, and hence AUC_Λ may be calculated. This follows from the relation

$$\Lambda = \frac{\mathrm{pr}(\Lambda|H_1)}{\mathrm{pr}(\Lambda|H_0)}, \tag{6}$$

which implies that the density function in the numerator is determined by the one in the denominator. There are, in fact, explicit integral expressions for AUC_Λ in terms of $M_0(\beta)$.[5,6] The function $M_0(\beta)$ satisfies $M_0(0) = M_0(1) = 1$. This motivates the definition of the likelihood-generating function $G(\beta)$ given by

$$M_0(\beta) = \exp[\beta(\beta - 1)G(\beta - \tfrac{1}{2})]. \tag{7}$$

This function, too, completely determines the statistics of Λ under both hypotheses.[5] The value of this function at the origin is given directly by

$$\begin{aligned} G(0) &= -4 \ln\left[M_0\left(\frac{1}{2}\right)\right] \\ &= -4 \ln\left[\int_0^\infty [\mathrm{pr}(\Lambda|H_1)\mathrm{pr}(\Lambda|H_0)]^{1/2}\mathrm{d}\Lambda\right] \\ &= -4 \ln\left[\int_0^\infty [\mathrm{pr}(\mathbf{g}|H_1)\mathrm{pr}(\mathbf{g}|H_0)]^{1/2}\mathrm{d}\mathbf{g}\right] \\ &= -4\mathrm{d}_B, \end{aligned} \tag{8}$$

where d_B is the Bhattacharyya distance between the two densities. This number seems to have some significance for estimating AUC_Λ. It will appear in the bounds that we discuss below and can also be used in the approximation

$$2(1 - \mathrm{AUC}_\Lambda) \approx 1 - \mathrm{erf}\{[\tfrac{1}{2}G(0)]^{1/2}\}, \tag{9}$$

which has proved to be accurate in most of the cases examined so far.[5,6] This is called the $\mathrm{SNR}_{G(0)}$ approximation in analogy with the familiar approximation

$$2(1 - \mathrm{AUC}_T) \approx 1 - \mathrm{erf}[\tfrac{1}{2}\mathrm{SNR}_T], \tag{10}$$

which is often used to relate AUC_T to the ordinary signal-to-noise (SNR) ratio for an arbitrary test statistic T. By this analogy, the quantity $[2G(0)]^{1/2}$ is called $\mathrm{SNR}_{G(0)}$. One of the motivations for this work is to find quantitative expressions for the accuracy of the $\mathrm{SNR}_{G(0)}$ approximation.

A. Known Bounds on AUC_Λ

It will be convenient to state our results and those of others in terms of the quantity D_Λ defined by

$$D_\Lambda = -\ln[2(1 - \mathrm{AUC}_\Lambda)]. \tag{11}$$

We have $0 \leqslant D_\Lambda < \infty$, and D_Λ is a monotonically increasing function of AUC_Λ. It was shown elsewhere[5] that

$$\tfrac{1}{2}G(0) \leqslant D_\Lambda. \tag{12}$$

If P_e is the probability of error under equally likely hypotheses, then

$$\begin{aligned} \min_{\Lambda_0}\{\mathrm{FPF}(\Lambda_0) + \mathrm{FNF}(\Lambda_0)\} &= 1 + \mathrm{FPF}(1) - \mathrm{TPF}(1) \\ &= 2P_e \end{aligned} \tag{13}$$

(where FNF is the false-negative fraction). The quantity on the left is the minimum total error rate. Shapiro used the inequalities

$$1 - \{1 - \exp[-\tfrac{1}{2}G(0)]\}^{1/2} \leqslant 2P_e \leqslant \exp[-\tfrac{1}{4}G(0)] \tag{14}$$

and

$$-\ln(2P_e) \leqslant D_\Lambda \leqslant -2\ln(2P_e) \tag{15}$$

to derive

$$\tfrac{1}{4}G(0) \leqslant \Delta_\Lambda \leqslant G(0) + 2\ln(1 + \{1 - \exp[-\tfrac{1}{2}G(0)]\}^{1/2}). \tag{16}$$

Notice that the lower bound on AUC_Λ is not an improvement over relation (12), although Shapiro did show that the lower bound in relation (15) can be.

If we set

$$\mu(\beta) = \log M_0(\beta) = \beta(\beta - 1)G(\beta - \tfrac{1}{2}), \tag{17}$$

then bounds derived by Burnashev[2] for the minimum total error rate may be written as

$$\begin{aligned} \mu(\beta) + (\tfrac{1}{2} - \beta)\mu'(\beta) - \tfrac{1}{2}[(\mu'(\beta))^2 + \mu''(\beta)]^{1/2} \\ \leqslant \ln(2P_e) \leqslant \mu(\beta) \end{aligned} \tag{18}$$

for $0 \leqslant \beta \leqslant 1$. By using relation (15) and setting $\beta = 1/2$, we can get

$$\begin{aligned} \tfrac{1}{4}G(0) &\leqslant D_\Lambda \\ &\leqslant \tfrac{1}{2}G(0) + \{2G(0) + \tfrac{1}{16}[G'(0)]^2 - \tfrac{1}{4}G''(0)\}^{1/2} \end{aligned} \tag{19}$$

B. New Upper Bound for AUC_Λ

We can improve on this last upper bound by doing a little more work. From the definition of AUC_Λ we can show that

$$\begin{aligned} 2(1 - \mathrm{AUC}_\Lambda) = \int_0^\infty \int_0^\infty \min(\Lambda_1, \Lambda_2)\mathrm{pr}(\Lambda_1|H_0) \\ \times \mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2 \end{aligned} \tag{20}$$

and

$$\begin{aligned} 2(1 - \mathrm{AUC}_\Lambda) = \int_0^\infty \int_0^\infty \exp\left[-\frac{1}{2}\left|\ln\left(\frac{\Lambda_2}{\Lambda_1}\right)\right|\right](\Lambda_1\Lambda_2)^{1/2} \\ \times \mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2. \end{aligned} \tag{21}$$

We can see that the second expression follows from the first by noting that

$$\exp\left[-\frac{1}{2}\left|\ln\left(\frac{\Lambda_2}{\Lambda_1}\right)\right|\right](\Lambda_1\Lambda_2)^{1/2} = \min(\Lambda_1, \Lambda_2). \tag{22}$$

To prove the first [Eq. (20)] we have

$$\int_0^\infty\int_0^\infty \min(\Lambda_1,\Lambda_2)\mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2$$
$$= \int_0^\infty\int_0^{\Lambda_2} \Lambda_1\mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2 + \int_0^\infty\int_{\Lambda_2}^\infty \Lambda_2\mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2$$
$$= \int_0^\infty\int_0^{\Lambda_2} \Lambda_1\mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2 + \int_0^\infty\int_0^{\Lambda_1} \Lambda_2\mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_2\mathrm{d}\Lambda_1$$
$$= 2\int_0^\infty\int_0^{\Lambda_1} \Lambda_2\mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_2\mathrm{d}\Lambda_1$$
$$= 2\int_0^\infty\int_0^{\Lambda} \mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_1)\mathrm{d}\Lambda_2\mathrm{d}\Lambda_1$$
$$= 2(1 - \mathrm{AUC}_\Lambda). \tag{23}$$

It is interesting to note that we can also use Eq. (20) to show that

$$2(1 - \mathrm{AUC}_\Lambda) = \int_0^\infty [\mathrm{FPF}(\Lambda)]^2\mathrm{d}\Lambda, \tag{24}$$

but we will not be using this particular expression below.[6]

Now we can follow a derivation similar to Burnashev's to get

$$-\mu(\beta) - \mu(1-\beta) \leqslant D_\Lambda \leqslant -2\mu(\beta) - (1-2\beta)\mu'(\beta) + \tfrac{1}{2}[2\mu''(\beta)]^{1/2}. \tag{25}$$

for $0 \leqslant \beta \leqslant 1$. At $\beta = \frac{1}{2}$ this gives

$$\tfrac{1}{2}G(0) \leqslant D_\Lambda \leqslant \tfrac{1}{2}G(0) + [G(0) - \tfrac{1}{8}G''(0)]^{1/2}. \tag{26}$$

Notice that the lower bound here is the same as in relation (12) and that the quantity under the square root is less than half of the corresponding quantity in relation (19). These bounds are therefore tighter than those in relation (19).

Here is the derivation. Let

$$p_\beta(\Lambda) = \frac{\Lambda^\beta \mathrm{pr}(\Lambda|H_0)}{M_0(\beta)}, \tag{27}$$

and note that this is a nonnegative function that integrates to 1. The function $p_\beta(\Lambda)$ may therefore be regarded as a probability density on Λ. Expectations with respect to this density will be indicated by the subscript β on angle brackets.

The following expression is equivalent to Eq. (21):

$$2(1-\mathrm{AUC}_\Lambda) = M_0(\alpha)M_0(\beta)\int_0^\infty\int_0^\infty \exp\left[-\frac{1}{2}\left|\ln\left(\frac{\Lambda_2}{\Lambda_1}\right)\right|\right] \times \Lambda_1^{1/2-\alpha}\Lambda_2^{1/2-\beta}p_\alpha(\Lambda_1)p_\beta(\Lambda_2)\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2$$
$$= M_0(\alpha)M_0(\beta)\left\langle \exp\left[-\frac{1}{2}\left|\ln\left(\frac{\Lambda_2}{\Lambda_1}\right)\right|\right]\Lambda_1^{1/2-\alpha}\Lambda_2^{1/2-\beta}\right\rangle_{\alpha,\beta}. \tag{28}$$

Now we use the fact that for any random variable x, $\ln\langle x\rangle \geqslant \langle \ln x\rangle$ (Jensen inequality) and $\langle x\rangle^2 \leqslant \langle x^2\rangle$ (Schwarz inequality) to get a lower bound for the logarithm of the left-hand side of Eq. (28):

$$\ln[2(1-\mathrm{AUC}_\Lambda)] \geqslant \mu(\alpha) + \mu(\beta) - \left\langle \frac{1}{2}\left|\ln\left(\frac{\Lambda_2}{\Lambda_1}\right)\right|\right\rangle_{\alpha,\beta} + \left\langle\left(\frac{1}{2}-\alpha\right)\ln\Lambda_1\right\rangle_{\alpha,\beta} + \left\langle\left(\frac{1}{2}-\beta\right)\ln\Lambda_2\right\rangle_{\alpha,\beta}$$
$$\geqslant \mu(\alpha) + \mu(\beta) - \frac{1}{2}\left[\left\langle \ln^2\left(\frac{\Lambda_2}{\Lambda_1}\right)\right\rangle_{\alpha,\beta}\right]^{1/2} + \left\langle\left(\frac{1}{2}-\alpha\right)\ln\Lambda_1\right\rangle_{\alpha,\beta} + \left\langle\left(\frac{1}{2}-\beta\right)\ln\Lambda_2\right\rangle_{\alpha,\beta}. \tag{29}$$

We can evaluate each of the terms appearing in the last line. They are given by

$$\langle(\tfrac{1}{2}-\alpha)\ln\Lambda_1\rangle_{\alpha,\beta} = (\tfrac{1}{2}-\alpha)\mu'(\alpha), \tag{30}$$

$$\langle(\tfrac{1}{2}-\beta)\ln\Lambda_2\rangle_{\alpha,\beta} = (\tfrac{1}{2}-\beta)\mu'(\beta), \tag{31}$$

and

$$\langle(\ln\Lambda' - \ln\Lambda)^2\rangle_{\alpha,\beta} = \mu''(\alpha) + [\mu'(\alpha)]^2 - 2\mu'(\alpha)\mu'(\beta) + \mu''(\beta) + [\mu'(\beta)]^2. \tag{32}$$

(Note that $\mu'(\beta)$ is the derivative of $\mu(\beta)$ and $\mu''(\beta)$ is the second derivative.) Using these relations and setting $\alpha = \beta$ leads to

$$\ln[2(1-\mathrm{AUC}_\Lambda)] \geqslant 2\mu(\beta) + (1-2\beta)\mu'(\beta) - \tfrac{1}{2}[2\mu''(\beta)]^{1/2}. \tag{33}$$

If we set $\beta = \frac{1}{2}$, then the first-derivative term vanishes, and we have

$$\ln[2(1-\mathrm{AUC}_\Lambda)] \geqslant 2\mu(\tfrac{1}{2}) - \tfrac{1}{2}[2\mu''(\tfrac{1}{2})]^{1/2}.$$

In terms of $G(0)$ this is

$$\ln[2(1-\mathrm{AUC}_\Lambda)] \geqslant -\tfrac{1}{2}G(0) - \tfrac{1}{2}[4G(0) - \tfrac{1}{2}G''(0)]^{1/2}. \tag{34}$$

To get the other side of relation (26) is easier. In fact, it has already been proved by other techniques in Ref. 5. Here we will start with Eq. (20) and make use of the restriction $0 \leqslant \beta \leqslant 1$:

$$
\begin{aligned}
2(1 - \mathrm{AUC}_\Lambda) &= \int_0^\infty \int_0^\infty \min\left[\frac{\Lambda_1}{\Lambda_2}, 1\right] \Lambda_2 \mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1 \mathrm{d}\Lambda_2 \\
&\leqslant \int_0^\infty \int_0^\infty \left(\frac{\Lambda_1}{\Lambda_2}\right)^\beta \Lambda_2 \mathrm{pr}(\Lambda_1|H_0)\mathrm{pr}(\Lambda_2|H_0)\mathrm{d}\Lambda_1 \mathrm{d}\Lambda_2 \\
&= M_0(\beta)M_0(1 - \beta). \qquad (35)
\end{aligned}
$$

By taking logarithms we have

$$\ln[2(1 - \mathrm{AUC}_\Lambda)] \leqslant \mu(\beta) + \mu(1 - \beta). \qquad (36)$$

Setting $\beta = 1/2$ finally leads to

$$\ln[2(1 - \mathrm{AUC}_\Lambda)] \leqslant 2\mu(\tfrac{1}{2}) = -\tfrac{1}{2}G(0). \qquad (37)$$

3. EXAMPLE 1. INDEPENDENT EXPONENTIAL

A simplistic model for speckle noise is that the components of $\mathbf{g}$ are independent, exponentially distributed random variables. This would correspond to small, widely spaced detectors viewing a speckle pattern. A more realistic model would try to incorporate correlations among the components of $\mathbf{g}$, but that complication will not be considered here. If the signal changes the mean data vector from $\mathbf{b}$ to $\mathbf{b} + \mathbf{s}$, then

$$\mathrm{pr}(\mathbf{g}|H_1) = \prod_{m=1}^{M} \frac{1}{b_m} \exp\left(-\frac{g_m}{b_m}\right), \qquad (38)$$

$$\mathrm{pr}(\mathbf{g}|H_2) = \prod_{m=1}^{M} \frac{1}{b_m + s_m} \exp\left(-\frac{g_m}{b_m + s_m}\right). \qquad (39)$$

We are assuming that $\mathbf{b}$ and $\mathbf{s}$ are vectors with positive components. If the signal had a component that was zero, then the corresponding detector would not contribute anything to the likelihood ratio. This means that M represents the number of detectors that are influenced by the signal. This assumption about the background and signal will also be made in the second example below. If we introduce the normalized quantities $\alpha_m = b_m(b_m + s_m)^{-1}$ and $\gamma_m = s_m(b_m + s_m)^{-1}$, then the moment-generating function for λ when the signal is absent is

$$M_0(\beta) = \prod_{m=1}^{M} \frac{\alpha_m^\beta}{1 - \gamma_m \beta}. \qquad (40)$$

To investigate the relation between the bounds given above and the actual values for AUC_Λ, we consider the case where $s_m = s$ and $b_m = b$ for all m. If we let $\delta = s/b$, then the exact value for AUC_Λ is given by[6]

$$\mathrm{AUC}_\Lambda = \left(\frac{1 + \delta}{2 + \delta}\right)^M \sum_{k=0}^{M-1} \frac{(M + k - 1)!}{k!(M - 1)!} \left(\frac{1}{2 + \delta}\right)^k. \qquad (41)$$

The $\mathrm{SNR}_{G(0)}$ approximation is given by

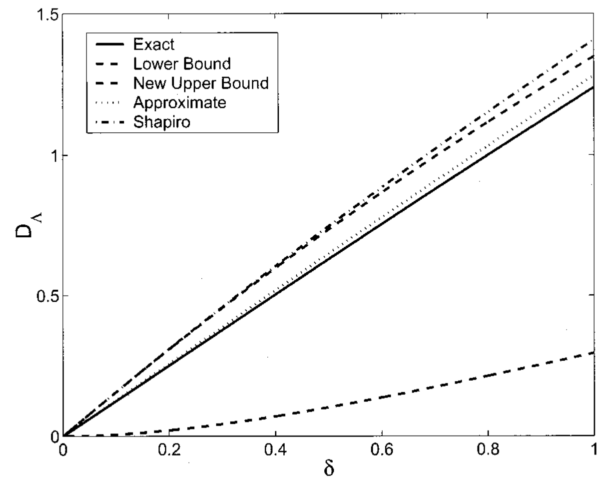

Fig. 1. Exact D_Λ, approximate D_Λ from relation (9), upper bound from relation (16), and both bounds from relation (26) versus the contrast parameter δ for independent exponential noise with a flat background ($M = 5$).

$$2(1 - \mathrm{AUC}_\Lambda) \approx 1 - \mathrm{erf}\left(\left\{\frac{M}{2} \ln\left[\frac{(2 + \delta)^2}{4(1 + \delta)}\right]\right\}^{1/2}\right). \qquad (42)$$

The bounds in relation (26) for this example are

$$\frac{M}{2} \ln\left[\frac{(2 + \delta)^2}{4(1 + \delta)}\right] \leqslant D_\Lambda \leqslant \frac{M}{2} \ln\left[\frac{(2 + \delta)^2}{4(1 + \delta)}\right] + \frac{\delta(2M)^{1/2}}{2 + \delta}, \qquad (43)$$

and the upper bound in relation (16) is given by

$$D_\Lambda \leqslant M \ln\left[\frac{(2 + \delta)^2}{4(1 + \delta)}\right] + 2 \ln\left(1 + \left\{1 - \left[\frac{4(1 + \delta)}{(2 + \delta)^2}\right]^{M/2}\right\}^{1/2}\right). \qquad (44)$$

These bounds together with the exact and the approximate values for AUC_Λ are shown in Fig. 1 as functions of δ.

If we try to get a better bound by minimizing the right-hand side of relation (25) with respect to β, we find that the minimum occurs at

$$\tilde{\beta} = \frac{1}{2}\left[1 + \left(\frac{1}{2M}\right)^{1/2}\right]. \qquad (45)$$

For large M this number is very close to 1/2, and therefore we would not expect to gain much from this attempt. Similarly, if we try to maximize the left-hand side in relation (25), we find that it occurs at $\beta = 1/2$.

4. EXAMPLE 2. POISSON

Poisson statistics are important in imaging problems where the number of photons collected is relatively small. If the signal changes the mean from $\mathbf{b}$ to $\mathbf{b} + \mathbf{s}$, then

$$\mathrm{pr}(\mathbf{g}|H_1) = \prod_{m=1}^{M} \frac{\exp[g_m \log(b_m) - b_m]}{g_m!}, \qquad (46)$$

$$\mathrm{pr}(\mathbf{g}|H_2) = \prod_{m=1}^{M} \frac{\exp[g_m \log(b_m + s_m) - b_m - s_m]}{g_m!}. \qquad (47)$$

The moment-generating function for λ under the signal-absent hypothesis is

$$M_0(\beta) = \exp\left(-\beta \sum_{m=1}^{M} s_m\right) \exp\left[\sum_{m=1}^{M} b_m(\kappa_m^{\beta} - 1)\right], \tag{48}$$

with $\kappa_m = (b_m + s_m)b_m^{-1}$.

If we again examine the case where $b_m = b$ and $s_m = s$ for all m, we find that the exact AUC_Λ is given by[6]

$$\mathrm{AUC}_\Lambda = 1 - \exp[-Mb(1 + \kappa)]\left[\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(Mb)^{k+l}\kappa^l}{(1 + \delta_{kl})k!l!}\right]. \tag{49}$$

(Note that δ_{kl} is the Kronecker delta symbol in this expression.) The $\mathrm{SNR}_{G(0)}$ approximation is given by

$$2(1 - \mathrm{AUC}_\Lambda) \approx 1 - \mathrm{erf}\,[(Mb)^{1/b}(\sqrt{\kappa} - 1)]. \tag{50}$$

The bounds from relation (26) are now

$$Mb(\sqrt{\kappa} - 1)^2 \leqslant D_\Lambda Mb(\sqrt{\kappa} - 1)^2 + \frac{\ln \kappa}{2}(2Mb\sqrt{\kappa})^{1/2}, \tag{51}$$

and the upper bound in relation (16) is

$$D_\Lambda \leqslant 2Mb(\sqrt{\kappa} - 1)^2 + 2\ln(1 + \{1 - \exp[-Mb(\sqrt{\kappa} - 1)^2]\}^{1/2}). \tag{52}$$

These bounds, together with the exact and approximate AUC_Λ are shown in Fig. 2 as functions of κ and in Fig. 3 as functions of Mb.

The minimum value of the right-hand side of relation (25) as a function of β occurs at the number $\tilde{\beta}$ that satisfies

$$(1 - 2\tilde{\beta})\kappa^{\tilde{\beta}/2} = \frac{1}{2(2Mb)^{1/2}}. \tag{53}$$

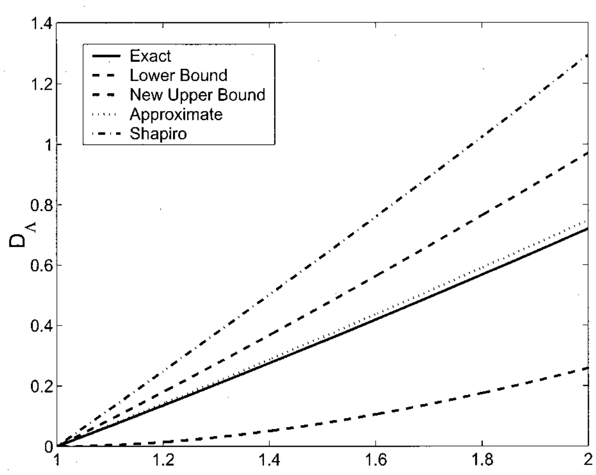

Fig. 2. Exact D_Λ, approximate D_Λ from relation (9), upper bound from relation (16), and both bounds from relation (26) versus the contrast parameter κ for independent Poisson noise with a flat background ($Mb = 1.5$).

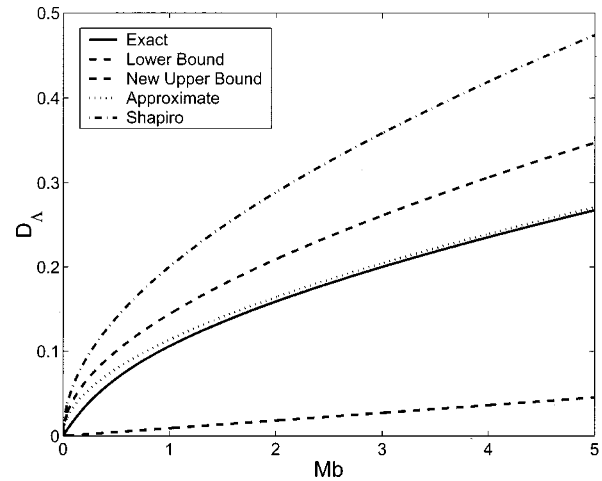

Fig. 3. Exact D_Λ, approximate D_Λ from relation (9), upper bound from relation (16), and both bounds from relation (26) versus total mean photon count Mb for independent Poisson noise with a flat background ($\kappa = 1.2$).

Again we find that for large M, this number is close to 1/2. In fact, an approximation for large M is given by

$$\tilde{\beta} \approx \frac{1}{2}\left[1 - \frac{1}{2(2Mb\sqrt{\kappa})^{1/2}}\right]. \tag{54}$$

As in Example 1, there is not much to gain in the upper bound by using this value instead of 1/2 for β. The maximum value of the left-hand side of relation (25) is at $\beta = 1/2$ for this example also.

5. ASYMPTOTIC RESULTS

In general, the minimization of the upper bound in relation (25) leads to the equation

$$(1 - 2\tilde{\beta})[2\mu''(\tilde{\beta})]^{3/2} = \mu'''(\tilde{\beta}). \tag{55}$$

When Λ is the product of M independent, identically distributed random variables, as it is when the data are independent and identically distributed, then $\mu(\beta) = M\mu_1(\beta)$, where $\mu_1(\beta)$ is the logarithm of the moment-generating function for the case $M = 1$. This is the case for the two examples given above. The equation for $\tilde{\beta}$ is now

$$(1 - 2\tilde{\beta})[2\mu_1''(\tilde{\beta})]^{3/2} = \frac{1}{\sqrt{M}}\mu_1'''(\tilde{\beta}). \tag{56}$$

This shows that $\tilde{\beta}$ will always be close to 1/2 in this situation when M is large.

We also have $G(\beta) = MG_1(\beta)$, which gives

$$1 \leqslant \frac{D_\Lambda}{\frac{1}{2}MG_1(0)} \leqslant 1 + \frac{2}{G_1(0)\sqrt{M}}\left[G_1(0) - \frac{1}{8}G_1'''(0)\right]^{1/2}. \tag{57}$$

This in turn implies that

$$\lim_{M \to \infty}\left[\frac{D_\Lambda}{\frac{1}{2}MG_1(0)}\right] = 1 \tag{58}$$

Notice that this result would not follow from the other two upper bounds given above.

We may use our approximate expression for AUC_Λ in relation (9) to approximate D_Λ:

$$\hat{D}_\Lambda = -\ln(1 - \mathrm{erf}\,\{[\tfrac{1}{2}G(0)]^{1/2}\}). \tag{59}$$

A standard inequality for the error function gives us

$$\left|\frac{\hat{D}_\Lambda}{\frac{1}{2}MG_1(0)} - 1\right| \leqslant \frac{2}{MG_1(0)}\ln\left(\frac{\sqrt{\pi}}{2}\left\{\left[\frac{1}{2}MG_1(0)\right]^{1/2} + \left[\frac{1}{2}MG_1(0) + 2\right]^{1/2}\right\}\right), \tag{60}$$

which leads to the asymptotic result

$$\lim_{M\to\infty}\left[\frac{\hat{D}_\Lambda}{\frac{1}{2}MG_1(0)}\right] = 1. \tag{61}$$

Combining these two limits we then have

$$\lim_{M\to\infty}\left[\frac{D_\Lambda}{\hat{D}_\Lambda}\right] = 1. \tag{62}$$

This result is of some interest in imaging where M is the number of detectors and is typically a very large number. However, the assumption that the detector outputs are independent and identically distributed can apply only if we are trying to distinguish between flood images of different intensities. For more realistic tasks the detector outputs will be neither independent nor identically distributed. A generalization of these asymptotic results to that situation would provide a useful check for ideal-observer calculations. If we have a method for estimating the likelihood ratio, then D_Λ and $\hat{D}_\Lambda$ can be estimated by Monte Carlo methods. If they do not agree, then we would suspect that our estimate for the likelihood ratio is not very good.

6. CONCLUSION

A new upper bound for the AUC of the ideal observer in a signal-detection task has been found. It has also been shown how this bound, together with a lower bound originally derived elsewhere, leads to asymptotic approximations for this AUC when the data are independent and identically distributed. A future goal is to extend these asymptotic results to more general situations. The search will also continue for tighter bounds on the ideal-observer AUC in order to constrain system optimization problems that use this quantity as a figure of merit.

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